Continuous Time Dynamic Models

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References

- Doraszelski and Judd (2012): less computation in continuous than discrete time
- Estimation and identification: Arcidiacono et al. (2016), Blevins (forthcoming)
- Applications:
    - Schiraldi, Smith, and Takahashi (2013)
    - Cosman (2014)

Comparing continuous and discrete time models

- See discussion in Doraszelski and Judd (2012)
- Move order matters – e.g. Cournot vs Stackelberg competition
- Discrete time model limits how often and how much state variables can change
- Embedding problem: sometimes there does not exist a continuous time Markov chain that induces the same probability distribution over states at discrete times as a discrete time Markov chain
- Often no compelling reason to prefer a discrete or continuous time model, but important to remember that they do have slightly different assumptions and implications

Why continuous time reduces computation

- Discrete time simultaneous move game suffers from "curse of dimensionality" in computing expectations
  - E.g. entry/exit game with N firms has at least 2^N possible states next period
- When only one player could move each instant then number of possible future states is much lower
- Continuous time: assume move opportunities arrive stochastically, then $\mathbb{P}(\text{two move at same time}) = 0$

Section 1

Model

Model

- Notation of Arcidiacono et al. (2016)
- N players indexed by i
- Finite state space $\mathcal{X}$ with K elements, indexed by k
- J actions in $\mathcal{A} = \{0, \ldots, J-1\}$.
- Flow payoff u_{ik} from being in state k
- Instantaneous payoff $\psi_{ijk} + \epsilon_{ij}$ from choosing j in state k
- Discount rate ρ

State Transitions

- States follow an exogenous Markov jump process with intensity matrix:

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1K} \\ \vdots & \ddots & \vdots \\ q_{K1} & \cdots & q_{KK} \end{bmatrix}$$

where

$$q_{kl} = \lim_{h \to 0} \frac{\mathrm{P}(X_{t+h} = l | X_t = k)}{h}$$

is the rate of arrival of moves to state l given state k.

- States also change from actions: $l(m, j, k) =$ state after player m chooses j in state k

Strategies

- Moves arrive at rate λ
- Beliefs of player $\zeta_{imjk} = \mathrm{P}($ player m chooses j in state $k)$
- Value function:

$$V_{ik}(\zeta_i) = \frac{u_{ik} + \sum_{l \neq k} V_{il}(\zeta_i) + \sum_{m \neq i} \lambda \sum_j \zeta_{imjk} V_{i,l(m,j,k)}(\zeta(i)) + \lambda \mathrm{E}[\max_j \psi_{ijk} + \epsilon_{ij} + V_{i,l(i,j,k)}(\zeta_i)]}{\rho + \sum_{l \neq k} q_{kl} + N\lambda}$$

- Best response choice probabilities

$$\sigma_{ijk} = \mathrm{P}(\psi_{ijk} + V_{i,l(i,j,k)}(\zeta_i) + \epsilon_{ij} \geq \psi_{ij'k} + V_{i,l(i,j',k)}(\zeta_i) + \epsilon_{ij'} \forall j')$$

- Equilibrium $\sigma_{-i} = \zeta_i$ for all i

Identification

- Argument is mostly similar to discrete time
- Q and choice probabilities are identified from observed distribution of states
    - Extra argument needed if observed data is at discrete intervals – see Arcidiacono et al. (2016) for details
- Given Q and knowing distribution of ϵ, differences in value functions are given by a known function of choice probabilities
- Expected (over other players actions) payoffs recovered from Bellman equation
- Exclusion identifies payoffs

Section 2

Estimation

Estimation

Step 1 : estimate hazards and choice probabilities

$$\hat{h} = \arg\max_h \sum_{m=1}^{M} \sum_{n=1}^{T} \underbrace{\log g(\tau_{mn}, k_{mn}; h)}_{\text{likelihood of waiting } \tau_{nm} \text{ to next event given state } k_{mn}} + \underbrace{\sum_{l \neq k_{mn}} I_{mn}(0,l) \log q_{k_{mn} l}}_{\text{next move exogenous state variable}} + \underbrace{\sum_i \sum_{j = \neq 0} I_{mn}(i,j) \log(\lambda \sigma_{ijk_{mn}})}_{\text{next move by a player}}$$

Estimation

Step 2 : given $\hat{h}$ compute best response choice probabilites, represent implied hazards as $\Lambda(\theta, \hat{h})$

$$\hat{\theta} = \arg\max_{\theta} \sum_{m=1}^{M} \sum_{n=1}^{T} g(\tau_{mn}, k_{mn}; \Lambda(\theta, \hat{h}) + \\ + \sum_{i} \sum_{j=\neq 0} I_{mn}(i,j) \log(\lambda \Lambda_{ijk_{mn}}(\theta, \hat{h}))$$

Section 3

Applications

Walmart's Entry into the Supermarket Industry

- Application of Arcidiacono et al. (2016)
- Data: Trade Dimensions Retail Database 1994-2006
- Market = MSA with population $\leq$ 500,000

TABLE 1

Summary statistics

	Mean	S.D.	Max.
Number of chains present[a]	2.559	0.024	7
Average No. of stores per chain[b]	3.727	0.040	32
Number of Walmarts present[a]	1.004	0.142	12
Number of fringe firms present[a]	12.997	0.823	47
Number of new chain stores[c]	0.277	0.012	5
Number of exiting chain stores	0.224	0.011	7
Number of new fringe stores	0.825	0.021	10
Number of exiting fringe stores	0.908	0.023	11
Number of new Walmarts	0.177	0.008	3
Number of exiting Walmarts	0.002	0.001	1
Population increase	0.042	0.004	1
Population decrease	0.004	0.001	1

[a] Sample size is 2910.

[b] Sample size is 7446 and removes all market-period combinations where the chain operates no stores.

[c] Sample size in this and all remaining rows is 2686.

TABLE 2

Response to initial Walmart entry

	Year Before	Year During	Year After
Number of new chain stores	0.311	0.211	0.189
	(0.064)	(0.054)	(0.041)
Number of exiting chain stores	0.122	0.156	0.189
	(0.038)	(0.044)	(0.050)
Number of new fringe stores	0.867	0.711	0.767
	(0.117)	(0.105)	(0.102)
Number of exiting fringe stores	0.789	0.844	0.833
	(0.114)	(0.118)	(0.132)

Notes: Standard errors in parentheses. Based on 90 markets where Walmart is first observed to enter.

Model

- 3 firm types: Walmart, Chain, & Fringe
- State: $x_k = (\overbrace{s^c_{1k}, s^c_{2k}, \dots}^{\#\text{ stores of chain firms}}, \underbrace{s^f_{1k}, s^f_{2k}, \dots}_{\text{fringe firms}}, \overbrace{s^w_k}^{\text{Walmart}}, \underbrace{d_k}_{\text{population}})$

  and unobserved static market-type z
- Payoffs: flow profits partially cubic in x_k, plus opening, entry, and closing costs

TABLE 3
Chain firm parameters

	Time-aggregated		With Walmart entry times		No unobserved heterogeneity	
	Coeff.	S.E.	Coeff.	S.E.	Coeff.	S.E.
Constant (θ_0^c)	4.470	(0.768)	4.403	(0.749)	2.561	(0.409)
Number of chain stores (θ_1^c)	−0.065	(0.024)	−0.067	(0.024)	−0.017	(0.014)
Number of Walmarts (θ_2^c)	−0.375	(0.148)	−0.383	(0.139)	−0.278	(0.108)
Number of fringe stores (θ_3^c)	−0.052	(0.017)	−0.053	(0.017)	−0.040	(0.012)
Number of own stores (θ_4^c)	−0.039	(0.081)	−0.044	(0.084)	0.104	(0.051)
Number of own stores Sq./100 ($100 \times \theta_5^c$)	−0.182	(0.432)	−0.165	(0.445)	−0.265	(0.166)
Population (θ_6^c)	0.176	(0.114)	0.213	(0.111)	0.267	(0.075)
Unobserved state (θ_7^c)	−0.956	(0.881)	−0.968	(0.806)		
Unobserved state × number of own stores (θ_8^c)	0.245	(0.199)	0.249	(0.191)		
Entry cost (η_0^c)	−18.377	(0.805)	−18.400	(0.807)	−17.643	(0.953)
Entry cost × Unobserved State (η_1^c)	−5.151	(1.621)	−5.148	(1.676)		
Store building cost (κ_0^c)	−5.068	(0.876)	−5.073	(0.870)	−4.494	(0.782)
Store building cost × Unobserved State (κ_1^c)	3.513	(0.968)	3.508	(0.986)		
Exit value (ϕ_0^c)	15.913	(0.888)	15.912	(0.896)	15.044	(0.633)
Exit value × unobserved state (ϕ_1^c)	4.166	(1.261)	4.126	(1.274)		

TABLE 4

Fringe firm parameters

	Time-aggregated		With Walmart entry times		No unobserved heterogeneity	
	Coeff.	S.E.	Coeff.	S.E.	Coeff.	S.E.
Constant (θ_1^f)	−13.074	(0.080)	−13.092	(0.080)	−12.698	(0.067)
Number of chain stores (θ_2^f)	−0.021	(0.003)	−0.021	(0.003)	−0.018	(0.003)
Number of Walmarts (θ_2^f)	−0.041	(0.012)	−0.042	(0.012)	−0.054	(0.012)
Number of fringe stores (θ_3^f)	0.183	(0.008)	0.183	(0.008)	0.193	(0.008)
Number of fringe stores squared / 100 ($100 \times \theta_4^f$)	−0.349	(0.018)	−0.349	(0.019)	−0.369	(0.018)
Population (θ_5^f)	0.240	(0.021)	0.248	(0.021)	0.170	(0.021)
Unobserved state (θ_6^f)	−2.530	(0.107)	−2.544	(0.107)		
Unobserved state × number of fringe stores (θ_7^f)	0.050	(0.006)	0.051	(0.006)		
Entry cost (η_0^f)	−5.034	(0.033)	−5.034	(0.033)	−5.030	(0.033)
Entry cost × Unobserved State (η_1^f)	1.186	(0.079)	1.190	(0.079)		

TABLE 5

Counterfactual simulations of market structure in year 2014 with and without Walmart

	Markets	Initial Pop	Chain firms	Chain stores	Fringe stores	Walmart stores	Chain share (%)	Walmart share (%)	Fringe share (%)	C1 (%)	C3 (%)	HHI
With Walmart												
All Markets	205	17,6,153	2.41	9.17	11.98	2.42	39.9	10.8	49.4	25.4	48.1	0.22
Midwest	58	175,371	1.75	5.88	14.36	2.07	27.3	9.9	62.7	21.7	39.1	0.20
Northeast	22	205,180	2.18	8.48	14.32	2.58	35.2	10.7	54.1	24.0	45.3	0.21
South	83	170,856	2.78	11.72	9.63	2.85	49.1	12.1	38.7	29.1	55.8	0.24
West	42	172,494	2.71	9.02	12.11	1.96	41.3	9.3	49.4	23.8	46.6	0.20
Absent Walmart												
All Markets	205	176,153	2.77	12.43	9.85	0.00	54.9	0.0	44.6	29.9	55.7	0.26
Midwest	58	175,371	2.13	8.41	11.81	0.00	42.0	0.0	58.0	27.6	47.5	0.25
Northeast	22	205,180	2.61	12.22	11.18	0.00	53.7	0.0	46.3	30.6	55.4	0.27
South	83	170,856	3.22	16.15	7.54	0.00	66.9	0.0	32.7	33.2	64.3	0.28
West	42	172,494	2.86	10.77	11.02	0.00	49.5	0.0	48.8	25.9	50.2	0.23

TABLE 6

Counterfactual simulations of changes in market structure due to Walmart's presence

	Markets	Initial Pop	Walmart stores	Chain stores (%)	Fringe stores (%)	Chain share (%)	Fringe share (%)	C1 (%)	C3 (%)	HHI (%)
All markets	205	176,153	2.42	−26.3	21.6	−27.3	10.6	−15.0	−13.7	−16.6
By region										
Midwest	58	175,371	2.07	−30.1	21.6	−34.8	8.1	−21.3	−17.8	−19.3
Northeast	22	205,180	2.58	−30.5	28.1	−34.5	16.8	−21.5	−18.4	−21.8
South	83	170,856	2.85	−27.4	27.7	−26.5	18.7	−12.4	−13.2	−15.6
West	42	172,494	1.96	−16.3	9.9	−16.6	1.2	−8.1	−7.2	−11.9
By market size										
Small	104	117,740	1.76	−24.3	7.0	−23.1	5.0	−12.1	−9.9	−14.1
Large	101	236,300	3.09	−27.4	30.0	−31.7	16.2	−18.1	−17.9	−19.4
By growth type										
Slow	54	178,252	2.26	−35.8	40.6	−36.5	24.3	−22.3	−21.0	−23.8
Moderate	46	175,444	2.17	−38.9	18.5	−38.3	15.2	−16.3	−17.6	−9.7
Fast	105	175,383	2.61	−17.4	13.5	−18.6	1.0	−10.1	−8.4	−14.9
By unobserved type										
More negative	9	106,248	1.20	−18.5	−17.2	−6.7	−6.2	29.6	27.3	42.3
Negative	68	127,754	1.62	−15.9	2.7	−17.9	−2.5	−9.4	−5.5	−11.3
Zero	96	184,404	2.20	−27.8	25.2	−31.1	18.2	−20.0	−19.2	−22.0
Positive	32	273,906	5.08	−30.5	59.1	−33.7	40.6	−15.1	−17.8	−17.7

TABLE 7

Counterfactual simulations of changes in market structure absent unobserved heterogeneity

	Walmart stores	Chain stores (%)	Fringe stores (%)	Chain share (%)	Fringe share (%)	C1 (%)	C3 (%)	HHI (%)
All Markets	3.15	−33.2	7.3	−33.1	9.0	−12.9	−9.7	−7.0
				By region				
Midwest	2.92	−38.4	9.7	−41.5	8.3	−17.2	−12.0	−6.6
Northeast	3.24	−35.8	7.0	−36.0	9.7	−14.6	−10.6	−7.5
South	3.28	−32.3	7.4	−29.8	11.7	−10.0	−8.5	−6.6
West	3.15	−28.6	3.4	−29.4	3.9	−12.4	−8.7	−7.6
				By market size				
Large	3.65	−30.5	7.3	−29.6	7.2	−12.2	−9.5	−7.8
Small	2.66	−37.5	7.1	−36.2	11.5	−13.9	−10.3	−6.8
				By unobserved type				
More Negative	2.22	−42.9	15.6	−43.9	13.1	−16.6	−14.9	−2.9
Negative	2.71	−35.9	6.0	−36.9	8.5	−14.9	−9.8	−6.5
Zero	3.24	−33.7	6.7	−32.8	7.6	−12.2	−9.1	−6.2
Positive	4.07	−29.6	9.3	−24.4	13.7	−10.1	−10.1	−12.5

TABLE 8

Temporal evolution of market structure

Year	Market size	WM stores	Chain stores (%)	Fringe stores (%)	Chain share (%)	Fringe share (%)	C1 (%)	C3 (%)	HHI (%)
5	Small	0.58	−6.8	3.4	−7.9	−0.1	−5.7	−6.3	−9.9
5	Large	0.87	−10.0	5.2	−9.9	2.9	−5.2	−7.5	−10.1
10	Small	1.06	−13.7	5.4	−14.3	0.8	−9.3	−9.2	−14.3
10	Large	1.76	−16.8	12.8	−18.3	6.6	−10.2	−12.0	−15.6
15	Small	1.45	−19.5	6.5	−19.4	2.0	−11.5	−10.3	−15.4
15	Large	2.52	−22.3	21.3	−25.6	11.1	−14.6	−15.2	−18.2
20	Small	1.76	−24.3	7.0	−23.1	5.0	−12.1	−9.9	−14.1
20	Large	3.09	−27.4	30.0	−31.7	16.2	−18.1	−17.9	−19.4

Entertainment districts and the value of variety in nightlife: evidence from Chicago

- Competition between businesses in a set of closely related industries
- Structural model: infer consumer preferences, firm's problem from observing entry and exit
- Strong consumer preference from variety — entrant can raise incumbent profits
- High barriers to entry matter for nightlife supply

Related economic literature

- Consumption amenities and valuation of cities
  - Glaeser (2001), Rappaport (2008), Lee (2010), Albouy (2013)
- Measuring consumers' value of access to variety
  - Broda & Weinstein (2006), Consumer goods: Li (2012), Broda & Weinstein (2010), Handbury & Weinstein (2011), Couture (2014)
- Explaining colocation of similar businesses
  - Theoretical: Wolinsky (1983), Fischer & Harrington (1996), Konishi (2005)
  - Empirical: Davis (2006), Jia (2008), Dunne *et al.* (2013), Datta & Sudhir (2013), Yang (2014)
- Profit functions from entry/exit decisions
  - Bresnahan & Reiss (1991), Pesendorfer & Schmidt-Dengler (2003), Aguirregabiria and Mira (2007), Ryan (2012), Collard-Wexler (2013), Dunne *et al.* (2013)

Structural modelling approach

- Data on venue entry and exit — find parameters to rationalize as equilibrium

- Build model in stages:
  1. Static model: consumers choose to go out, venues choose price
  2. Dynamic model: venues choose whether to enter and exit
  3. Estimation: match parameters to observed entry and exit

- Static and dynamic counterfactuals

Static model

Consumer's problem

- Nested CES utility — substitution within, between venue types
- Reservation utility shock: stay in or go out?
- More utility to going out means more consumers choose to do so

Firm's problem

- Firms adjust prices to maximize profits taking into account consumer preferences, each others' behaviour
- Unique equilibrium prices for given number of competitors

Necessary assumption: interact only within neighbourhood

Dynamic model and continuous-time estimation

Dynamic model of entry and exit

- Entrants, incumbents receive opportunities via Poisson process
- Entrants can enter with given type, neighbourhood
- Best-respond to consistent beliefs — Markov-Nash equilibrium

Continuous-time structural estimation

- Arcidiacono, Bayer, Blevins, Ellickson
- Intuition: choose structural parameters so observed entry, exit rates rationalized as equilibrium
- Advantages: feasibility, data usage, flexibility

Data sources

Venues and regulation from City of Chicago Data Portal (2006–2014)

- Divide venues into categories based on licensing:
  1. Amusement only (e.g. Los Globos Ballroom)
  2. Drinks only (e.g. Casual Tap)
  3. Drinks and amusement (e.g. Tabu)
  4. Drinks and music (e.g. New Celebrity Lounge)

- Two types of within-city regulation:
  1. Dry areas: no bars at all
  2. Moratoria: no new bars

- Divide city into neighbourhoods based on community areas

Demographic data from Census, American Community Survey

Estimated preference for variety

Elasticity	Symbol	Estimate
Between sectors	η	2.04 (0.002)
Amusement only	ρ_1	4.90 (0.013)
Drinks only	ρ_2	2.15 (0.001)
Drinks and amusement	ρ_3	3.56 (0.224)
Drinks and music	ρ_4	7.96 (0.290)

- Amusement only, Drinks and amusement $\approx 5^{th} - 25^{th}$ percentile of consumer goods (Broda and Weinstein (2010))
- Drinks and music $\approx$ restaurants (Couture (2014))

Results: entry sunk cost and exit payoff

	Value (thousands of dollars)	
Entry cost	Amusement only baseline	862 [839, 886]
	Drinks only baseline	943 [871, 1023]
	Drinks and amusement baseline	892 [797, 995]
	Drinks and music baseline	670 [83, 7588]
Exit payoff	Amusement only	38.4 [36.6, 3383.7]
	Drinks only	38.3 [37.5, 39.8]
	Drinks and amusement	42.9 [36.8, 201.4]
	Drinks and music	40.5 [38.5, 44.3]

Barriers to entry

Is $700k-$900k to open a bar reasonable?

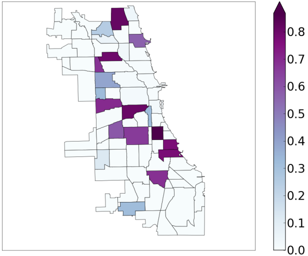
- Small business literature:
  - PowerHomeBiz: $239k-$837k depending on jurisdiction
  - Houston Chronicle: up to $1 million depending on licensing requirements
  - IBISWorld Industry Reports: $200k-$1 million
- Regulatory expenses: fees, time uncertainty, renovations to comply
- Marketing, hiring, cash on hand for payment systems

One more venue: impacts on profits

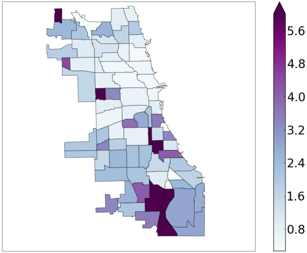
Percentage of observations where counterfactual new venue would *increase* incumbent profit

	Amusement only	Drinks only	Drinks and amusement	Drinks and music
Amusement only	36.3 [0.0,36.3]	13.2 [0.0,13.6]	6.7 [6.4,19.1]	14.1 [0.0,14.1]
Drinks only	13.3 [12.7,13.6]	13.2 [0,14.5]	17.8 [9.5,18.5]	8.4 [0.0,8.6]
Drinks and amusement	0.0 [0.0,0.3]	1.1 [0.0,1.2]	32.2 [0.0,86.8]	12.4 [0.0,12.4]
Drinks and music	0.0 [0.0,0.0]	1.1 [0.0,1.1]	13.3 [0.0,13.3]	25.3 [0.0,26.3]

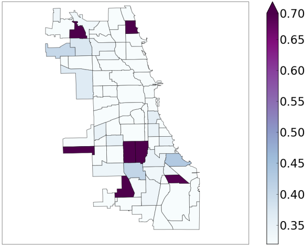
Dynamic counterfactual: lower barriers to entry



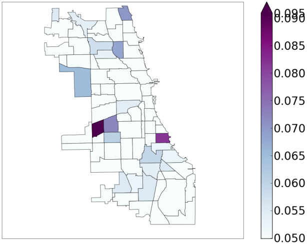
Amusement only



Drinks only



Drinks and amusement



Drinks and music

Discussion and further research

- Dynamic structural model for competition of related businesses
- Strong preferences for variety, high barriers to entry
- Further research: non-pecuniary benefits and goodness of fit

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